Department of Applied Mathematics

[illegible]

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[illegible] **[illegible]**

1. [illegible]

Formulate Newton's method for solving the nonlinear 2×2 system of equations

$$\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases}.$$

In the same style as how one proves quadratic convergence in the scalar case for $f(x)=0$ show quadratic convergence (assuming sufficient smoothness of f, g, root being simple, etc.) in the 2×2 case. Assuming the root $x=\alpha$, $y=\beta$ to be of multiplicity one, define $\varepsilon_n = x_n-\alpha$, $\eta_n = y_n-\beta$, and show that both ε_{n+1} and η_{n+1} are of size $O(\varepsilon_n^2\ \eta_n^2)$

2. [illegible]

Consider the quadrature formula

$$I_{quad} = \sum_{i=0}^{n} \alpha_i f(x_i) \qquad x_i \in [-1,1] \tag{1}$$

for the integral

$$I = \int_{-1}^{1} f(x)\, w(x)\, dx,$$

where $w(x)$

3. [illegible]

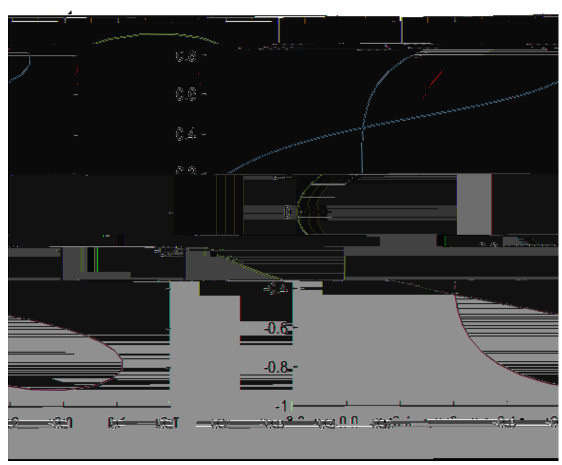
Assuming that φ_n, $n = 0, 1, 2, \ldots$ form a set of orthogonal polynomials of degrees *n* over some interval [*a*,*b*] with weight function *w*(*x*) > 0, show that they obey a three-term recursion relation of the form

$$\varphi_{n+1}(x) = (a_n x + b_n)\varphi_n(x) + c_n \varphi_{n-1}(x), \quad n = 1, 2, 3, \ldots$$

where the coefficients a_n, b_n, c_n do not depend on *x*.

4. Linear Algebra

Let $A \in \mathbb{C}^{n \times n}$ be a symmetric complex valued matrix, $A = A^T$. It is possible to show that one can find vectors u and nonnegative numbers μ solving the so-called



6. Numerical PDE

Consider the Poisson's equation

$$(\partial_{xx}+\partial_{yy})\,u = f(x,y), \quad (x,y)\in B = [0,1]\times[0,1]$$

with the Dirichlet boundary condition

$$u\big|_{(x,y)\in\partial B} = 0$$

Set f to be

$$f(x,y) = 4-3\pi^2)\,(4\;3\pi x\,(4\,]\quad \pi x \quad \pi y - \quad \pi y$$
$$-4\pi^2 3\;)\,(4\pi y\,)+(4\,]\quad \pi y \quad \pi x - \quad \pi x$$

yielding the solution

$$u(x,y) = (2\;)\,(4\;x)-(4\;)\quad \pi x \quad \pi y - \quad \pi y$$

At a first glance it may appear that seeking a solution as a sine series,

$$u(x,y) = \sum_{m,n=1}^{\infty} \hat{u}_{mn} \sin(\pi m x)\sin(\pi n y)$$

should be an efficient approach. However, it turns out that the sine series converges rather slowly.

(a) Can you figure out why the convergence of the sine series is fairly slow?

(b) What are other bases one can use to achieve high accuracy? Suggest a basis that would be more efficient in this case.

(c) Sketch a numerical scheme to compute the solution with high accuracy.