Applied Analysis: Preliminary Exam

Department of Applied Mathematics, University of Colorado at Boulder

10.00am { 1.00pm, August 17, 2010

Problem 1: Set = [1, 1] and define for 2 ( ) the operator via

$$[\quad](x) = -\frac{d}{dx}(1 \quad x^2)\frac{d}{dx}\ (x).$$

Set

$$\Omega = \{ \quad : \quad 2 \quad (\ )\}.$$

(a) Find a function 2 ( ) such that [ ](x) = x.

(b) Show that Ω ( ).

(c) For a function 2 Ω, give an explicit formula for a function 2 ( ) such that = . (Your formula may involve unevaluated integrals, and/or sums of unevaluated integrals.)

(d) Describe the topological closure $\overline{\Omega}$ of Ω in ( ). (For any 2 $\overline{\Omega}$, the equation = has a solution 2 ( ) when the di erential operator is defined in a \weak" sense.)

Hint for Problem 1: Define for = 0, 1, 2, 3, . . . the functions $_n$ via

(1)
$$_n(x) = \sqrt{\frac{2 \quad + 1}{2}}\ \frac{1}{2^n \quad !}\left(\frac{d}{dx}\right)^n (x \quad 1)^n.$$

You may use that

(2)
$$_n = \quad (\quad + 1) \quad _n,$$

and that $\{ \ _n\}_n^{\infty}$ is an orthonormal basis for ( ).

Problem 2: Specify which of the following statements are true. No justification necessary.

(a) The set of even functions is dense in ([1, 1]).

(b) The set of polynomials is dense in ([1, 1]).

(c) The set of simple functions is dense in ( ). (Recall that a *simple function* is a function of the form $= \sum_j^J \ _j\, \chi_{\Omega_j}$ where is a finite integer, $_j$ is a scalar, and Ω_j is a measurable subset of .)

(d) The set of bounded continuous functions is dense in $^{\infty}$( ).

(e) The set ([1, 1]) is dense in ([1, 1]).

(f) The space ^{p}( ) is separable for all p such that 1 $p < \infty$.

(g) The space $^p(\mathbb{N})$ is separable for all p such that 1 p ∞.

(h) The space ([1, 1]) is separable.