APPM 3570/STAT 3100 Exam 1 Spring 2023

NAME: ____

SECTION: 001 or 002

Instructions:

1. Calculators are permitted.
2. Notes, your text and other books, cell phones, and other electronic devices are not permitted | except for calculators or as needed to view and upload your work.
3. Justify your answers, show all work.
4. When you have completed the exam, go to the uploading area in the room and scan your exam and upload it to Gradescope.
5.

Problem 1. (20 points.) The following five questions pertain to permutations of the following letters: $x; y; y; z; z; z; v; v; v; v; w; w; w; w; w$. Do <u>not</u> simplify your answers.

(a) How many different permutations are there?

(b) How many permutations start with a w and end with an x?

(c) How many permutations keep identical letters together?

(d) How many permutations contain the sub-sequence $v; z; v; z; v; z; v$?

(e) How many permutations keep no two y's together?

Solution:

(a) (4 points.) $\binom{15}{1;2;3;4;5} = \frac{15!}{1!\,2!\,3!\,4!\,5!}$.

(b) (4 points.) $\binom{13}{2;3;4;4} = \frac{13!}{2!\,3!\,4!\,4!}$.

(c) (4 points.) 5!.

(d) (4 points.) $\binom{9}{1;2;1;5} = \frac{9!}{1!\,2!\,1!\,5!}$.

(e) (4 points.) $\binom{15}{1;2;3;4;5} \quad \binom{14}{1;1;3;4;5}$.

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Problem 2. (24 points.) There are three unrelated parts to this question.

(a) Four events occur with probabilities $P(E) = 0.35$; $P(F) = 0.15$; $P(G) = 0.40$; $P(B) = 0.30$. If $P(E|B) = 0.20$, what's the probability that exactly one of E or B occurs? Simplify your answer!

(b) A drawer has 8 forks, 8 knives, 4 spoons and one spatula. If I draw 10 objects randomly, what's the probability that I get at least one fork and one spatula? Do not simplify your answer.

(c) A fast-food Mexican restaurant sells burritos with a choice of up to seven different fillings. If customers are equally likely to ask for any combination of at least one and all the possible fillings, what's the probability a new customer asks for a burrito with all seven fillings? Simplify your answer!

Solution:

(a) (8 points.)

$$\begin{aligned} P(BE^c \cup EB^c) &= P(BE^c) + P(EB^c) \\ &= P(B) - P(EB) + P(E) - P(EB) \\ &= 0.65 - 2P(EB). \end{aligned}$$

But $P(EB) = P(E|B)P(B) = (0.2)(0.3) = 0.06$. So $P(BE^c \cup EB^c) = 0.65 - 0.12 = 0.53$.

(b) (8 points.)

P(at least one fork and one spatula are selected)

$= 1 -$ P(no forks are selected) $-$ P(no spatulas are selected) $+$ P(neither a fork nor a spatula is selected)

$$= 1 - \left[\frac{\binom{13}{10}}{\binom{21}{10}} + \frac{\binom{20}{10}}{\binom{21}{10}} - \frac{\binom{12}{10}}{\binom{21}{10}} \right].$$

(c) (8 points.)

$$\frac{1}{\sum_{k=1}^{7}\binom{7}{k}} = \frac{1}{\sum_{k=0}^{7}\binom{7}{k} - 1} = \frac{1}{\sum_{k=0}^{7}\binom{7}{k}1^k 1^{7-k} - 1} = \frac{1}{(1+1)^7 - 1} = \frac{1}{2^7 - 1} = \frac{1}{127}.$$

(Alternatively, $\sum_{k=1}^{7}\binom{7}{k} = 7 + 21 + 35 + 35 + 21 + 7 + 1 = 127$.)

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Problem 3. (24 points.) A manufacturer produces vehicle batteries, some of which are defective. Assume that the probability that a battery is defective is 0.10. There is an electronic test to determine if a battery is or not defective. When the electronic test is conducted on a defective battery, the probability that the electronic test will be positive (i.e., indicate that the battery is defective) is 0.9. Instead, when the electronic test is conducted on a non-defective battery, the probability that the electronic test will be positive is 0.1.

(a) If the electronic test is conducted on a randomly selected battery, what is the probability that the test is positive? Simplify your answer!

(b)

Problem 4. (32 points.) A biased coin is twice more likely to come up heads than tails. Let G be the number of heads minus the number of tails observed when the coin is tossed independently three times.

(a) What's the probability of flipping heads in one coin toss?

(b) Determine the probability mass function (p.m.f.) of G.

(c) Find $P(-1 \le G \le 2)$.

(d) Determine the cumulative distribution function (c.d.f.) of G.

Solution:

(a) (3 points.) $P(\text{"Heads"}) = 2 \cdot P(\text{"Tails"})$ but $P(\text{"Heads"}) + P(\text{"Tails"}) = 1$. So $P(\text{"Heads"}) = 2/3$.

(b) (12 points.) Observe that $G = H - (3 - H) = 2H - 3$, where H is the number of heads in the three flips. In particular, since $H \in \{0, 1, 2, 3\}$, $G \in \{-3, -1, 1, 3\}$. Futher

$$P(G = -3) = P(H = 0) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{1}{27};$$

$$P(G = -1) = P(H = 1) = 3\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{2}{9};$$

$$P(G = 1) = P(H = 2) = 3\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{4}{9};$$

$$P(G = 3) = P(H = 3) = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{8}{27}.$$

The p.m.f. of G is

$$P(G = g) = \begin{cases} 1/27, & \text{if } g = -3; \\ 2/9, & \text{if } g = -1; \\ 4/9, & \text{if } g = 1; \\ 8/27, & \text{if } g = 3; \\ 0, & \text{otherwise.} \end{cases}$$

(c) (5 points.)

$$P(-1 \le G \le 2) = P(G = -1 \text{ or } G = 1) = P(G = -1) + P(G = 1) = 2/9 + 4/9 = 2/3.$$

(d) (12 points.) From part (b), we find that the c.d.f. of G is

$$F_G(g) = P(G \le g) = \begin{cases} 0, & \text{for } g < -3; \\ 1/27, & \text{for } -3 \le g < -1; \\ 7/27, & \text{for } -1 \le g < 1; \\ 19/27, & \text{for } 1 \le g < 3; \\ 1, & \text{for } g \ge 3. \end{cases}$$

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Bonus Problem. (Recover up to 4 points marked down in problems 1-4.) Let A, B, and C be independent events. Are $(A \cup B)$ and C independent? Justify your answer with a mathematical argument or a counter-example.

Solution:

Solution I. Since $A; B; C$ are independent, $P(A|C) = P(A)$, $P(B|C) = P(B)$, and $P(AB|C) = P(AB)$. Hence, using the conditional version of the inclusion-exclusion formula:

$$\begin{aligned} P(A \cup B|C) &= P(A|C) + P(B|C) - P(AB|C) \\ &= P(A) + P(B) - P(AB) = P(A \cup B); \end{aligned}$$

so $A \cup B$ and C are independent.

Solution II. $A \cup B$ and C are independent because

$$\begin{aligned} P((A \cup B) \cap C) &= P(AC \cup BC) \\ &= P(AC) + P(BC) - P(AC \cap BC) \\ &= P(A) \cdot P(C) + P(B) \cdot P(C) - P(ABC) \\ &= P(A) \cdot P(C) + P(B) \cdot P(C) - P(A) \cdot P(B) \cdot P(C) \\ &= \left(P(A) + P(B) - P(A) \cdot P(B) \right) \cdot P(C) \\ &= \left(P(A) + P(B) - P(AB) \right) \cdot P(C) \\ &= P(A \cup B) \cdot P(C). \end{aligned}$$