

1. (42 pts) The following problems are unrelated.

(a) Find the derivative of $y = \frac{1}{5x^2} \sin x$.

(b) Evaluate $\int \frac{\arcsin(x)}{1-x^2} dx$.

(c) Evaluate $\int_0^{\ln 3} \frac{e^x}{1+e^{2x}} dx$.

(d) Estimate the value of $\int_1^5 \ln \frac{x}{x+1} dx$ using a Riemann sum with right endpoints and $n = 4$ rectangles of equal width. Express your answer in terms of a single logarithm.

(e) Evaluate $\lim_{x \rightarrow 1} 2x \sinh \frac{3}{x}$.

Solution:

(a) $\frac{dy}{dx} = \frac{10x \cos x}{2 \cdot 5x^2 \sin x}$:

(b) We will use the substitution $u = \arcsin(x)$. So, $du = \frac{1}{\sqrt{1-x^2}} dx$:

$$\begin{aligned} \int \frac{\arcsin(x)}{\sqrt{1-x^2}} dx &= \int u du \\ &= \frac{1}{2} u^2 + C \\ &= \frac{1}{2} (\arcsin(x))^2 + C \end{aligned}$$

(c) We will use the substitution $u = e^x$. So, $du = e^x dx$, the new upper limit of integration will be $u = e^{\ln 3} = 3$.

$$\begin{aligned}
\lim_{x \rightarrow 1} 2x \sinh \frac{3}{x} &= \lim_{x \rightarrow 1} \frac{2 \sinh \frac{3}{x}}{\frac{1}{x}} \\
&= \lim_{x \rightarrow 1} \frac{\cosh \frac{3}{x} \cdot \frac{6}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x \rightarrow 1} 6 \cosh \frac{3}{x} \\
&= 6:
\end{aligned}$$

2. (12 pts)

(a) State the definition of continuity of a function, $f(x)$, at a point, $x = a$:

(b) Now consider the function $f(x)$ defined on $[-1; 1]$ by

$$f(x) = \begin{cases} 2 \sin^{-1}(x) & \text{if } x < \frac{1}{2} \\ c & \text{if } x = \frac{1}{2} \\ \cos^{-1}(x) & \text{if } x > \frac{1}{2} \end{cases}$$

Is there a value of c that makes f continuous at $x = \frac{1}{2}$?

(a) We see that $h'(x) = \frac{1}{x} - \frac{1}{4}$. Since $h'(x)$ exists on the interior of the domain of $h(x)$ and $h'(x) = 0$ has only a solution of $x = 16$, then $x = 16$ is the only critical number of $h(x)$: Since $h'(x) > 0$ when $0 < x < 16$ and $h'(x) < 0$ when $x > 16$, then $h(x)$ has a local maximum value at $x = 16$, and no local minimum values.

(b) $r'(x) = 2 \tan(x) \sec^2(x) - \frac{1}{4} \tan(x) \sec^2(x)$:

4. (32 pts) Consider $s(x) = \frac{e^{2x}}{3 - e^{2x}}$.

(a) Determine $s'(1)$: (Your final answer should be in terms of e .)

(b) Determine the inverse of $s(x)$. Be sure to label your final answer as $s^{-1}(x)$: (You may assume without proof that $s(x)$ is one-to-one.)

(c) Determine all horizontal asymptotes of $s(x)$: Justify each with the appropriate limit.

(d) Determine all vertical asymptotes of $s(x)$: Justify each with the appropriate limit.

Solution:

Note: For many of these problems, you may alternatively note that

$$s(x) = \frac{e^{2x}}{3 - e^{2x}} = \frac{e^{2x}}{e^{-2x} - 1} = \frac{1}{3e^{-2x} - 1}$$

before proceeding. This will lead to solutions equivalent to the below.

(a)

$$\begin{aligned} s'(x) &= \frac{(3 - e^{2x})2e^{2x} - e^{2x}(-2e^{2x})}{(3 - e^{2x})^2} \\ &= \frac{6e^{2x}}{(3 - e^{2x})^2} \\ s'(1) &= \frac{6e^2}{(3 - e^2)^2} \end{aligned}$$

(b)

$$\begin{aligned} y &= \frac{e^{2x}}{3 - e^{2x}} \\ x &= \frac{e^{2y}}{3 - e^{2y}} \\ (3 - e^{2y})x &= e^{2y} \\ e^{2y}(x - 1) &= -3x \\ e^{2y} &= \frac{3x}{x - 1} \\ y &= \frac{1}{2} \ln \frac{3x}{x - 1} \\ s^{-1}(x) &= \frac{1}{2} \ln \frac{3x}{x - 1} \end{aligned}$$

(c) The following limits is an $\frac{1}{7}$

Solving for the desired rate and plugging in the known values at that moment, we have

$$\begin{aligned}\frac{d}{dt} &= \frac{\cos^2(\theta)}{9} \frac{dx}{dt} \\ &= \frac{(9-15)^2}{9} \cdot 3 \\ &= \frac{3}{25} \text{ radians per second:}\end{aligned}$$

Note: We used the Pythagorean theorem to find the length of the hypotenuse and determine $\cos \theta$ at that moment.